International Journal of Theoretical Physics, Vol. 18, No. 4, 1979

A New Approach to Einstein–Petrov Type I Spaces

S. B. Edgar

Mathematics Department, University of Cape Coast, Cape Coast, Ghana

Received November 15, 1978

The structure of the completely integrable system of the Einstein–Petrov type I equations (recently derived by Brans) is investigated. By introducing a modification to the Newman–Penrose tetrad formalism, not only is the complete system obtained very easily and presented very concisely, but it is possible to obtain a set of identities which give explicitly the considerable redundancy within the complete system. This redundancy is exploited to identify a very compact subsystem of the complete system which arises very naturally in the new formalism, and which is of a different (in some senses, simpler) structure than the usual presentations for these spaces. The usefulness of this new subsystem in the search for exact solutions is demonstrated.

1. INTRODUCTION

In a recent paper, Brans (1977) has investigated the integrability conditions of the Einstein–Petrov type I equations in a vacuum. He has shown that a nontrivial set of integrability conditions (the post-Bianchi equations) exist, and further that the integrability conditions of the post-Bianchi equations are identically satisfied modulo the other equations, i.e., the two sets of structure equations, plus the Bianchi equations, plus the post-Bianchi equations are a completely integrable system. The first part of Brans' work was carried out in differential form notation, but in the latter part he had to write out the results explicitly in the N.P. formalism. (Newman and Penrose, 1962). Although his calculations were extremely long and had to be carried out by computer, Brans emphasizes that the simple nature of the final result suggests that it should have been anticipated without the need for such explicit computations.

In the first part of this paper the structure of the completely integrable system determined by Brans is examined in detail. A modification to the N.P. formalism is introduced which, by simplifying the presentation of the Bianchi equations and highlighting certain aspects of their structure, enables the post-Bianchi equations to be easily found and the completely integrable system of equations to be identified and compactly presented. In addition the considerable redundancy within the complete system is easily determined and is also presented very concisely as a set of identities.

In the latter part of this paper this redundancy is exploited to obtain a subsystem of the complete system, which although a much smaller system and much more compact, is still sufficient to ensure that the complete system is satisfied. This subsystem has such a simple and natural presentation that it would suggest that there is still some underlying structure which remains to be fully appreciated and exploited. Further, the conciseness and simplicity of this new sufficient subsystem provides a very promising starting point in the search for new exact solutions; in particular, it is found that, for a subset of the Petrov type I spaces, the subsystem reduces very easily to three standard nonlinear differential equations.

Papapetrou (1971a, b) has obtained a number of fundamental results on the structure of tetrad formalisms, with particular emphasis on the N.P. formalism, and although his results were derived for only the most general case (completely arbitrary Riemann tensor), his methods are very relevant to this paper. His results are summarized in Section 2 along with their applications to the N.P. formalism. In Section 3 the new notation is introduced and the results on integrability discovered by Brans follow very easily. When the techniques of Papapetrou are applied to the complete system of equations for this special class of spaces, five sets of identities are found linking the equations of the system, and they are presented compactly in the new notation. These identities are given in Section 4.

In Section 5 it is shown (for the general Petrov type I spaces) that the redundancy found in Section 4 enables the first set of structure equations to be omitted from the complete system. Further it is seen that most of the equations in the second set of structure equations can also be omitted, and a very compact sufficient subsystem of equations is presented exclusively and naturally in the new notation. In Section 6 it is shown, for a subset of Petrov type I spaces, that the new presentation enables the crucial differential equations for the metric tensor to be obtained very efficiently, and a coordinate system is suggested naturally.

The results are summarized in Section 7 and their implications discussed. Appendix A contains an explicit statement of all the post-Bianchi equations in N.P. formalism, and the proof of a result used in Section 6, is given in Appendix B.

2. INTEGRABILITY CONDITIONS AND IDENTITIES FOR THE COMPLETELY ARBITRARY SPACES

The basic system of equations usually considered in the tetrad formalism are[1]

$$\gamma_{mnp} = Z_{ma;b} Z_n{}^a Z_p{}^b \tag{2.1}$$

$$R_{mnpq} = 2\gamma_{mn[p;q]} + 2\gamma_{sn[p}\gamma^s{}_{|m|q]} + 2\gamma_{mns}\gamma^s{}_{[pq]} \tag{2.2}$$

$$R_{mn[pq;r]} = \gamma_m{}^s{}_{[r} R_{pq]sn} - \gamma_n{}^s{}_{[r} R_{pq]sm} + 2R_{mns[p}\gamma_r{}^s{}_{q]} \tag{2.3}$$

Papapetrou (1971a) has emphasized the following important property: *The three sets of equations* (2.1), (2.2), *and* (2.3) *form a completely integrable system of equations.* [This is in the sense that when (2.1) is considered as a set of differential equations for $Z_m{}^a$, the integrability conditions are given by (2.2); when the set (2.2) is considered as a set of differential equations for γ_{mnp}, the integrability conditions are given by (2.3); when (2.3) is considered as a set of differential equations for R_{mnpq} the integrability conditions are satisfied identically in this case.]

Within this complete system there exists a lot of redundancy and it is easily identified in the notation introduced by Papapetrou (1971b). Using Papapetrou's notation, the complete system of equations can be presented as follows:

$$X_{mnp} = 0 \tag{2.1'}$$

$$Y_{mnpq} = 0 \tag{2.2'}$$

$$V_{sm[npq]} = 0 \tag{2.3'}$$

where

$$X_{mnp} \equiv 2\gamma_{[m|n|p]} - 2Z_{[m}{}^a{}_{;p]} Z_{na} \tag{2.4}$$

$$Y_{mnpq} \equiv R_{mnpq} - 2\gamma_{mn[p;q]} - 2\gamma_{sm[q}\gamma^s{}_{|n|p]} - 2\gamma_{mn}{}^s\gamma_{s[pq]} \tag{2.5}$$

$$V_{smnpq} \equiv R_{smnp;q} - 2R_{smtn}\gamma^t{}_{pq} + R_{pqtm}\gamma^t{}_{sn} - R_{pqts}\gamma^t{}_{mn} \tag{2.6}$$

The X_{mnp}, Y_{mnpq}, V_{smnpq} are merely labels for the different equations.

The redundancy is then given explicitly by the following three sets of identities:

$$X_{[m}{}^s{}_{n;p]} - X_r{}^s{}_{[m} X_n{}^r{}_{p]} + X_{[m}{}^r{}_n\{\gamma_{|r|}{}^s{}_{p]} - \gamma_{p]}{}^s{}_r\} + 2X_r{}^s{}_{[m}\gamma_n{}^r{}_{p]} + Y_{[m}{}^s{}_{np]} = 0 \tag{2.7}$$

[1] The orthonormal tetrad vectors are denoted by $Z_m{}^a$, the Ricci spin coefficients by γ_{mnp}, and the tetrad components of the Riemann tensor by R_{mnpq}. In general, the Latin letters in the latter half of the alphabet, $m, n, p, \ldots$, will be used for tetrad components of an object, while the letters at the beginning of the alphabet, $a, b, c, \ldots$, will be used for coordinate components. All indices run 1, 2, 3, 4. The covariant derivative and the intrinsic derivative will be denoted by a semicolon, while the ordinary partial derivative will be denoted by a comma. Antisymmetrization will be denoted by square brackets, symmetrization by round brackets, and the alternating tensor by η^{abcd}.

$$V_{mn[pqr]} - Y_{mn[pq;r]} + \gamma_{mn[p;|s|}X_q{}^s{}_{r]} + 2\gamma^s{}_{[pq}Y_{|mns|r]}$$
$$- \gamma^s{}_{m[p}Y_{|sn|qr]} + \gamma^s{}_{n[p}Y_{|sm|qr]} + \gamma_{mn}{}^s Y_{s[pqr]} = 0 \quad (2.8)$$

$$\eta^{pqrs}\{V_{mnpqr;s} + 3\gamma^t{}_{rs}V_{mn[pqt]} - 2\gamma^t{}_{[m|p}V_{t|n]qrs}$$
$$- R_{mntr}Y^t{}_{pqs} + R_{qrt[n}Y^t{}_{m]ps} + \tfrac{1}{2}R_{mnpq;t}X_r{}^t{}_s\} = 0 \quad (2.9)$$

A fourth set of equations often used in work in the tetrad formalism is the set of equations

$$\{\nabla_m\nabla_n + \gamma_{[m}{}^p{}_{n]}\nabla_p\}\eta = 0 \quad (2.10)$$

where η can be either completely arbitrary or represent two complex independent quantities. It is important to note that the commutator identities

$$\{\nabla_{[a}\nabla_{b]}\}\eta = 0 \quad (2.11)$$

become in tetrad notation,

$$\{\nabla_{[m}\nabla_{n]} + Z_{[m;n]}\nabla_a\}\eta = 0 \quad (2.12)$$

which becomes by virtue of (2.4),

$$\{\nabla_{[m}\nabla_{n]} + \gamma_{[m}{}^p{}_{n]}\nabla_p\}\eta = \tfrac{1}{2}X_m{}^p{}_n\nabla_p\eta \quad (2.13)$$

Another important result, also implicit in Papapetrou's work (1971a), is now obvious: *The equations* (2.1) *and* (2.10) *are equivalent.*

The N.P. formalism is simply the normalized tetrad formalism with specific choices made for the tetrad vectors (Newman and Penrose, 1962). Two of the tetrad vectors $Z_1{}^a$, $Z_2{}^a$ are chosen to be real, null, future-pointing vectors, normalized as

$$Z_1{}^a Z_{2a} = 1 \quad (2.14a)$$

and the other two vectors $Z_3{}^a$, $Z_4{}^a$ are chosen as complex null vectors and normalized as

$$Z_3{}^a Z_{4a} = -1 \quad (2.14b)$$

The 24 real rotation coefficients can be combined to give 12 complex spin coefficients and the Riemann tensor written as 12 independent complex components. A different symbol is given to each spin coefficient, differential operator, and independent Riemann tensor component. The four sets of equations (2.1), (2.2), (2.3), and (2.10) can easily be written out individually in this notation and it is obvious that all those results obtained above carry directly over into the N.P. formalism. It is also possible to apply these results to the G.H.P. formalism (originally suggested by Geroch, Held, and Penrose, 1973) where some useful modifications can be made (Edgar, 1978).

It is emphasized again that these results of Papapetrou were determined for the most general cases (i.e., tetrad and Riemann tensor completely arbitrary). In fact, Papapetrou (1970) originally considered only the vacuum case (Ricci tensor zero, but all other conditions arbitrary) and his results for the vacuum case are exactly what would be deduced by substituting a zero

Ricci tensor in the results above. So it is known that no extra integrability conditions or identities arise out of imposing the vacuum case constraint, and the above results can be carried over into this special case. However, when any other specializations are made it would be expected that the above results may need to be modified—as of course is clearly the situation for Petrov type I vacuum spaces.

3. INTEGRABILITY CONDITIONS FOR PETROV TYPE I SPACES

Brans (1977) commenced his work in differential form notation and then changed to the N.P. formalism. It will be convenient here to use the general tetrad formalism for conciseness but specialize to the N.P. formalism when explicit statements of individual equations are required.

Within the N.P. formalism there is still the tetrad freedom of (i) null rotations about $Z_1{}^a$, $Z_2{}^a$, and (ii) boost in $Z_1{}^a$-$Z_2{}^a$ plane, spatial rotation in $Z_3{}^a$-$Z_4{}^a$ plane. It is possible, for Petrov type I spaces, to use up this freedom in such a way that

$$\Psi_1 = 0 = \Psi_3 \tag{3.1a}$$

$$\Psi_0 = \Psi_4 \tag{3.1b}$$

so that the Weyl tensor is in canonical form.

Important symmetry properties are emphasized at this stage. It is well known that the tetrad equations for the general case are symmetric under the transformation (denoted by ′)

$$1 \leftrightarrow 2$$
$$3 \leftrightarrow 4$$

and also under the transformation (denoted by *)

$$1 \to 3 \qquad 3 \to -1$$
$$2 \to -4 \qquad 4 \to 2$$

(Both these symmetries are quoted by Geroch, Held, and Penrose, 1973, and in fact the ′ symmetry is built into the G.H.P. formalism.) It is interesting to note that the specialization (3.1) does not break these symmetries. In practice, this means that calculations can be considerably shortened. Unfortunately, the much more concise G.H.P. formalism cannot itself be used in the work, because by (3.1) a specific choice of gauge has been made.

The two sets of structure equations for vacuum Petrov type I spaces

$$\begin{aligned} X_{mnp} &\equiv 2\gamma_{[m|n|p]} - 2Z_{[m}{}^{a}{}_{;p]}Z_{na} \\ &= 0 \end{aligned} \tag{3.2}$$

$$\begin{aligned} Y_{mnpq} &\equiv R_{mnpq} - 2\gamma_{mn[p;q]} - 2\gamma_{sm[q}\gamma^{s}{}_{|n|p]} - 2\gamma_{mn}{}^{s}\gamma_{s[pq]} \\ &= 0 \end{aligned} \tag{3.3}$$

when written in the N.P. formalism become, respectively,

$$
\begin{aligned}
\kappa &= Z_{1a;b}Z_3{}^aZ_1{}^b, & \nu &= Z_{4a;b}Z_2{}^aZ_2{}^b\\
\rho &= Z_{1a;b}Z_3{}^aZ_1{}^b, & \mu &= Z_{4a;b}Z_2{}^aZ_4{}^b\\
\sigma &= Z_{1a;b}Z_3{}^aZ_4{}^b, & \lambda &= Z_{4a;b}Z_2{}^aZ_3{}^b\\
\tau &= Z_{1a;b}Z_3{}^aZ_2{}^b, & \pi &= Z_{4a;b}Z_2{}^aZ_1{}^b
\end{aligned}
$$

$$
\begin{aligned}
\beta &= \tfrac{1}{2}\{Z_{1a;b}Z_2{}^aZ_3{}^b - Z_{3a;b}Z_4{}^aZ_3{}^b\}\\
\alpha &= \tfrac{1}{2}\{Z_{1a;b}Z_2{}^aZ_4{}^b - Z_{3a;b}Z_4{}^aZ_4{}^b\}\\
\epsilon &= \tfrac{1}{2}\{Z_{1a;b}Z_2{}^aZ_4{}^b - Z_{3a;b}Z_4{}^aZ_1{}^b\}\\
\gamma &= \tfrac{1}{2}\{Z_{1a;b}Z_2{}^aZ_2{}^b - Z_{3a;b}Z_4{}^aZ_2{}^b\}
\end{aligned}
\tag{3.2$'$}
$$

$$
\begin{aligned}
-Y_{1341} &\equiv D\rho - \bar{\delta}\kappa - (\rho^2 + \sigma\bar{\sigma}) - (\epsilon + \bar{\epsilon})\rho + \bar{\kappa}\tau + \kappa(3\alpha + \bar{\beta} - \pi)\\
&= 0\\
-Y_{1331} &\equiv D\sigma - \delta\kappa - (\rho + \bar{\rho})\sigma - (3\epsilon - \bar{\epsilon})\sigma + (\tau - \bar{\pi} + \bar{\alpha} + 3\beta) - \Psi_0\\
&= 0\\
-Y_{2414} &\equiv D\lambda - \bar{\delta}\pi - (\rho\lambda + \bar{\sigma}\mu) - \pi^2 - (\alpha - \bar{\beta})\pi + \nu\bar{\kappa} + (3\epsilon - \bar{\epsilon})\lambda\\
&= 0\\
-Y_{2413} &\equiv D\mu - \delta\pi - (\bar{\rho}\mu + \sigma\mu) - \pi\bar{\pi} + (\epsilon + \bar{\epsilon})\mu + \pi(\bar{\alpha} - \beta) + \nu\kappa - \Psi_2\\
&= 0\\
-Y_{1321} &\equiv D\tau - \Delta\kappa - (\tau + \bar{\pi})\rho - (\bar{\tau} + \pi)\sigma - (\epsilon - \bar{\epsilon})\tau + (3\gamma + \bar{\gamma})\kappa\\
&= 0\\
-Y_{2412} &\equiv D\nu - \Delta\pi - (\pi + \bar{\tau})\mu - (\bar{\pi} + \tau)\lambda - (\gamma - \bar{\gamma})\pi + (3\epsilon + \bar{\epsilon})\nu\\
&= 0\\
-Y_{2424} &\equiv \Delta\lambda - \bar{\delta}\nu + (\mu + \bar{\mu})\lambda + (3\gamma + \bar{\gamma})\lambda - (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu + \Psi_0\\
&= 0\\
-Y_{1343} &\equiv \delta\rho - \bar{\delta}\sigma - \rho(\bar{\alpha} + \beta) + \sigma(3\alpha - \bar{\beta}) - (\rho - \bar{\rho})\tau - (\mu - \bar{\mu})\kappa\\
&= 0\\
-Y_{2434} &\equiv \delta\lambda - \bar{\delta}\mu - (\rho - \bar{\rho})\nu - (\mu - \bar{\mu})\pi - \mu(\alpha + \bar{\beta}) - \lambda(\bar{\alpha} - 3\beta)\\
&= 0\\
-Y_{2432} &\equiv \delta\nu - \Delta\mu - (\mu^2 + \lambda\bar{\lambda}) + (\gamma + \bar{\gamma})\mu + \bar{\nu}\pi - (\tau - 3\beta - \bar{\alpha})\nu\\
&= 0\\
-Y_{1323} &\equiv \delta\tau - \Delta\sigma - (\mu\sigma + \bar{\lambda}\rho) - (\tau + \beta - \bar{\alpha})\tau + (3\gamma - \bar{\gamma})\sigma + \kappa\bar{\nu}\\
&= 0\\
-Y_{1342} &\equiv \Delta\rho - \bar{\delta}\tau + (\rho\bar{\mu} + \sigma\lambda) - (\bar{\beta} - \alpha - \bar{\tau})\tau - (\gamma + \bar{\gamma})\rho - \nu\kappa + \Psi_2\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
-\tfrac{1}{2}\{Y_{1241} - Y_{3441}\} &\equiv D\alpha - \bar{\delta}\epsilon - (\rho + \bar{\epsilon} - 2\epsilon)\alpha - \beta\bar{\sigma} + \bar{\beta}\epsilon + \kappa\lambda\\
&\quad + \bar{\kappa}\gamma - (\epsilon + \rho)\pi\\
&= 0\\
-\tfrac{1}{2}\{Y_{1231} - Y_{3431}\} &\equiv D\beta - \delta\epsilon - (\alpha + \pi)\sigma - (\bar{\rho} - \bar{\epsilon})\beta + (\mu + \gamma)\kappa\\
&\quad + (\bar{\alpha} - \bar{\pi})\epsilon\\
&= 0
\end{aligned}
$$

$$\begin{aligned}
-\tfrac{1}{2}\{Y_{1221} - Y_{3421}\} &\equiv D\gamma - \Delta\epsilon - (\tau + \bar{\pi})\alpha - (\bar{\tau} + \pi)\beta + (\epsilon + \bar{\epsilon})\gamma \\
&\quad + (\gamma + \bar{\gamma})\epsilon - \tau\pi + \nu\kappa - \Psi_2 \\
&= 0 \\
-\tfrac{1}{2}\{Y_{1243} - Y_{3443}\} &\equiv \delta\alpha - \bar{\delta}\beta - (\mu\rho - \lambda\sigma) - \alpha\bar{\alpha} - \beta\bar{\beta} + 2\alpha\beta \\
&\quad - \gamma(\rho - \bar{\rho}) - \epsilon(\mu - \bar{\mu}) + \Psi_2 \\
&= 0 \\
-\tfrac{1}{2}\{Y_{1223} - Y_{3423}\} &\equiv \delta\gamma - \Delta\beta - (\tau - \bar{\alpha} - \beta)\gamma + (\gamma - \bar{\gamma} - \mu)\beta \\
&\quad - \mu\tau + \sigma\nu + \epsilon\bar{\nu} - \alpha\bar{\lambda} \\
&= 0 \\
-\tfrac{1}{2}\{Y_{1242} - Y_{3442}\} &\equiv \Delta\alpha - \bar{\delta}\gamma - (\rho + \epsilon)\nu + (\tau + \beta)\lambda \\
&\quad - (\bar{\gamma} - \bar{\mu})\alpha - (\bar{\beta} - \bar{\tau})\gamma \\
&= 0 \qquad (3.3')
\end{aligned}$$

It should be noted that the specialization (3.1) cannot be incorporated directly into the set of equations (3.3) but only into each individual equation when written out explicitly as in (3.3′). But since the simplification (3.1) does not affect the actual structure of the set (3.3) it will often be convenient to use (3.3), remembering that the substitution (3.1) is understood, rather than deal with the more lengthy (3.3′).

However, when the substitution (3.1) is made into the Bianchi identities (2.2) the basic structure is considerably changed as is clear in the N.P. formalism:

$$\bar{\delta}\Psi_0 = 3\kappa\Psi_2 + (4\alpha - \pi)\Psi_0 \qquad (3.4a)$$

$$\delta\Psi_0 = -3\nu\Psi_2 - (4\beta - \tau)\Psi_0 \qquad (3.4b)$$

$$\Delta\Psi_0 = 3\sigma\Psi_2 + (4\gamma - \mu)\Psi_0 \qquad (3.4c)$$

$$D\Psi_0 = -3\lambda\Psi_2 - (4\epsilon - \rho)\Psi_0 \qquad (3.4d)$$

$$D\Psi_2 = 3\rho\Psi_2 - \lambda\Psi_0 \qquad (3.5a)$$

$$\Delta\Psi_2 = -3\mu\Psi_2 + \sigma\Psi_0 \qquad (3.5b)$$

$$\delta\Psi_2 = 3\tau\Psi_2 - \nu\Psi_0 \qquad (3.5c)$$

$$\bar{\delta}\Psi_2 = -3\pi\Psi_2 + \kappa\Psi_0 \qquad (3.5d)$$

To use (2.2) as a concise statement of (3.4,5) is clearly unsatisfactory, and it is necessary to modify the notation in order to be able to state the equations (3.4,5) in a concise manner which displays their basic structure.

The spin coefficients are now relabeled as follows:

$$\begin{aligned}
a_m &= (\rho, -\mu, \tau, -\pi) \\
b_m &= (-\lambda, \sigma, -\nu, \kappa) \qquad (3.6) \\
c_m &= (-\epsilon, \gamma, -\beta, \alpha)
\end{aligned}$$

and so the Bianchi equations (3.4,5) can be written in the concise form

$$V_m{}^I \equiv \nabla_m \Psi_2 - 3a_m \Psi_2 - b_m \Psi_0 = 0 \tag{3.7}$$

$$V_m^{II} \equiv \nabla_m \Psi_0 - 3b_m \Psi_2 - \{a_m + 4c_m\} \Psi_0 = 0 \tag{3.8}$$

It is now easy to determine the integrability conditions for Ψ_2 and Ψ_0, and they are given in this notation as follows:

$$\begin{aligned} W_{mn}^I &\equiv 3\{a_{[m;n]} - \gamma_{[m}{}^p{}_{n]} a_p\} \Psi_2 \\ &\quad + \{b_{[m;n]} - \gamma_{[m}{}^p{}_{n]} b_p + 2a_{[m} b_{n]} + 4b_{[m} c_{n]}\} \Psi_0 \\ &= 0 \end{aligned} \tag{3.9}$$

$$\begin{aligned} W_{mn}^{II} &\equiv 3\{b_{[m;n]} - \gamma_{[m}{}^p{}_{n]} b_p + 4c_{[m} b_{n]} + 2b_{[m} c_{n]}\} \Psi_2 \\ &\quad + \{a_{[m;n]} - \gamma_{[m}{}^p{}_{n]} a_p + 4c_{[m;n]} - 4\gamma_{[m}{}^p{}_{n]} c_p\} \Psi_0 \\ &= 0 \end{aligned} \tag{3.10}$$

The $V_m{}^I$, V_m^{II}, W_{mn}^I, W_{mn}^{II}, X_{mnp}, Y_{mnpq} are here used merely as labels for the equations. The twelve equations (3.9), (3.10) are given explicitly in Appendix A.

When equations (3.9, 10) are examined closely it is clear that the only integrability conditions for a_m, b_m, c_m will arise from equations containing terms in $a_{[m;np]}$, $b_{[m;np]}$, $c_{[m;np]}$, i.e.,

$$W_{[m;np]}^I = 0 \tag{3.11}$$

$$W_{[m;np]}^{II} = 0 \tag{3.12}$$

When the equations (3.11, 12) are expanded and the commutators (2.17), the Bianchi equations (3.7, 8) and the post-Bianchi equations (3.9, 10) substituted, it is found that the equations (3.11, 12) are identically satisfied. Hence it can be concluded that all the eight integrability conditions arising from the set of differential equations (3.9, 10) are identically satisfied.

However, it is the whole differential system of equations which is under consideration and there is clearly the possibility of some inconsistency when the equations (3.3) are introduced alongside (3.9, 10). This inconsistency can be either algebraic or differential.

So the next step is to consider all the differential equations for the spin coefficients—(3.3) together with (3.9, 10)—and check for algebraic consistency between the two sets. In the above analysis it was convenient to change from the usual notation for the spin coefficients, and although it is not possible to rewrite the set of equations (3.3) completely in the new notation, it will be best to continue working in this notation. Since interest is mainly in the differential terms in equation (3.3) it is enough at the moment to note how these terms appear in the new notation for each equation:

$$\begin{array}{llll}
Y_{1342} : (a_{1;2} - a_{3;4}) & Y_{2431} : (a_{2;1} - a_{4;3}) \\
Y_{2442} : (b_{1;2} - b_{3;4}) & Y_{1331} : (b_{2;1} - b_{4;3}) \\
Y_{1341} : (a_{1;1} - b_{4;4}) & Y_{2432} : (a_{2;2} - b_{3;3}) \\
Y_{1323} : (a_{3;3} - b_{2;2}) & Y_{2414} : (a_{4;4} - b_{1;1}) \\
Y_{1343} : (a_{1;3} - b_{2;4}) & Y_{1321} : (a_{3;1} - b_{4;2}) \\
Y_{2434} : (a_{2;4} - b_{1;3}) & Y_{2412} : (a_{4;2} - b_{3;1})
\end{array}$$
$$\begin{array}{l}
\tfrac{1}{2}(Y_{1231} - Y_{3431}) : 2c_{[1;3]} \\
\tfrac{1}{2}(Y_{1224} - Y_{3424}) : 2c_{[2;4]} \\
\tfrac{1}{2}(Y_{1223} - Y_{3423}) : 2c_{(2;3)} \\
\tfrac{1}{2}(Y_{1214} - Y_{3414}) : 2c_{(1;4)} \\
\tfrac{1}{2}(Y_{1221} - Y_{3421}) : 2c_{(1;2)} \\
\tfrac{1}{2}(Y_{1243} - Y_{3443}) : 2c_{(3;4)}
\end{array} \tag{3.13}$$

It is clear from the form of the above expressions that the equations from the set (3.3) which will have to be tested for algebraic consistency against equations (3.9, 10) are the following:

$$Y_{1342} - Y_{2431} = 0 \tag{3.14a}$$
$$Y_{2442} - Y_{1331} = 0 \tag{3.14b}$$
$$Y_{1343} - Y_{1321} = 0 \tag{3.14c}$$
$$Y_{2434} - Y_{2412} = 0 \tag{3.14d}$$
$$Y_{1224} - Y_{3424} = 0 \tag{3.14e}$$
$$Y_{1231} - Y_{3431} = 0 \tag{3.14f}$$

When (3.14a, b) are written out explicitly and put into the new notation they become, respectively,

$$a_{[1;2]} - \gamma_{[1}{}^{p}{}_{2]}a_p - a_{[3;4]} + \gamma_{[3}{}^{p}{}_{4]}a_p = 0 \tag{3.15a}$$
$$b_{[1;2]} - \gamma_{[1}{}^{p}{}_{2]}b_p + 2a_{[1}b_{2]} + 4b_{[1}c_{2]} + \gamma_{[3}{}^{p}{}_{4]}b_p - 2a_{[3}b_{4]} - 4b_{[3}c_{4]} = 0 \tag{3.15b}$$

A comparison of these two equations with (3.9, 10) shows that there is no inconsistency; in fact, it means that two equations, W^{I}_{12} and W^{I}_{34}, of the six equations (3.9) [considered together with (3.15a, b)] are not independent, and so in practice one of these equations need not be included in the complete system. When (3.14c, d) and (3.14e, f) are written out explicitly in the new notation they become

$$a_{[1;3]} - \gamma_{[1}{}^{p}{}_{3]}a_p - b_{[2;4]} + \gamma_{[2}{}^{p}{}_{4]}b_p - 2b_{[2}a_{4]} - 4c_{[2}b_{4]} = 0 \tag{3.15c}$$
$$a_{[2;4]} - \gamma_{[2}{}^{p}{}_{4]}a_p - b_{[1;3]} + \gamma_{[1}{}^{p}{}_{3]}b_p - 2b_{[1}a_{3p} - 4c_{[1}b_{3]} = 0 \tag{3.15d}$$
$$c_{[2;4]} - \gamma_{[2}{}^{p}{}_{4]}c_p - a_{[1}b_{3]} - 2b_{[1}c_{3]} = 0 \tag{3.15e}$$
$$c_{[1;3]} - \gamma_{[1}{}^{p}{}_{3]}c_p - a_{[2}b_{4]} - 2b_{[2}c_{4]} = 0 \tag{3.15f}$$

A comparison of these equations with (3.9, 10) again reveals that there is no inconsistency. In fact, it means that there are two more pairs of equations in (3.9, 10)—W^I_{13} with W^{II}_{24} and W^I_{24} with W^{II}_{13}—which are not independent and so one equation from each pair can be excluded from the post-Bianchi equations.

So to sum up as regards algebraic consistency: it has been shown that all the equations in the sets (3.3) and (3.9, 10) are consistent, and further, that three of the equations in (3.9, 10) are not independent when considered alongside (3.3).

The final step is to check whether the introduction of (3.3) alongside (3.9, 10) will give rise to any new integrability conditions. First consider only equation (3.9) with (3.3) and it is clear from the form of the differential terms in (3.13) that the only equations which could possibly combine with (3.9) to give rise to integrability conditions are Y_{1342}, Y_{2424}, Y_{2413}, and Y_{1331}. But further investigation quickly reveals that the algebraic consistency just discussed eliminates the possibility of any integrability conditions. Secondly, consider only equation (3.10) with (3.3), and a comparison of the forms of the differential terms in c_m in (3.13) with the terms $c_{[m;n]}$ in (3.10) reveals that it is not possible to obtain new integrability conditions. This failure to be able to construct integrability conditions for the c_m alone clearly means that no new integrability conditions can exist when (3.9) and (3.10) together are considered alongside (3.3).

So to sum up, as regards differential consistency the introduction of (3.3) alongside (3.9, 10) does not add any more integrability conditions to those eight conditions (already found to be identically satisfied) which are associated with the equations (3.9, 10) alone.

Therefore it can be concluded that *the equations* (3.2), (3.3), (3.7, 8), *and* (3.9, 10) *form a completely integrable system of equations.*

This is of course exactly the result obtained by Brans but it has been derived here by a method which, as well as being very concise, makes the basic structure more transparent.

4. IDENTITIES FOR PETROV TYPE I SPACES

In order to determine the redundancy within the complete system of equations (3.2), (3.3), (3.7, 8), and (3.9, 10), consider the system of equations

$$X_m{}^p{}_n = 2\gamma_{[m}{}^p{}_{n]} - 2Z_{[m;}{}^a{}_{n]}Z^p{}_a \tag{4.1}$$

$$Y_{mnpq} = R_{mnpq} - 2\gamma_{mn[p;q]} + 2\gamma_{sm[p}\gamma^s{}_{|n|q]} - 2\gamma_{mns}\gamma^s{}_{[pq]} \tag{4.2}$$

$$V_m^I = \nabla_m\Psi_2 - 3a_m\Psi_2 - b_m\Psi_0 \tag{4.3}$$

$$V_m^{II} = \nabla_m\Psi_0 - 3b_m\Psi_2 - \{a_m + 4c_m\}\Psi_0 \tag{4.4}$$

$$W^I_{mn} = 3\Psi_2\{a_{[m;n]} - \gamma_{[m}{}^p{}_{n]}a_p\} + \Psi_0\{b_{[m;n]} - \gamma_{[m}{}^p{}_{n]}b_p + 4b_{[m}c_{n]} + 2a_{[m}b_{n]}\} \tag{4.5}$$

$$W^{II}_{mn} = 3\Psi_2\{b_{[m;n]} - \gamma_{[m}{}^p{}_{n]}b_p + 4c_{[m}b_{n]} + 2b_{[m}a_{n]}\} + \Psi_0\{a_{[m;n]} - \gamma_{[m}{}^p{}_{n]}a_p + 4c_{[m;n]} - 4\gamma_{[m}{}^p{}_{n]}c_p\} \tag{4.6}$$

remembering that the commutator identities are given by

$$\{\nabla_{[m}\nabla_{n]} + \gamma_{[m}{}^p{}_{n]}\nabla_p\}\eta - \tfrac{1}{2}X_m{}^p{}_n\nabla_p = 0 \tag{4.7}$$

The inhomogeneous system of equations (4.1)–(4.6) are now checked for consistency in the same way that the corresponding homogeneous system (3.2), (3.3), (3.7, 8) and (3.9, 10) has just been checked.

When (4.1) and (4.2) are considered as differential equations in $Z_m{}^a$ and γ_{mnp}, respectively, their respective integrability conditions are found to be

$$X_{[p}{}^s{}_{m;n]} - X_r{}^s{}_{[p}X_m{}^r{}_{n]} + X_{[p}{}^r{}_m\{\gamma_{|r|}{}^s{}_n - \gamma_{n]}{}^s{}_r\} + 2X_r{}^s{}_{[p}\gamma_m{}^r{}_{n]} + Y_{[p}{}^s{}_{mn]} = 0 \tag{4.8}$$

$$V_{sm[npq]} - Y_{sm[np;q]} + \gamma_{sm[n;|r|}X_p{}^r{}_{q]} - 2\gamma^r{}_{[np}Y_{|sm|q]r} - \gamma^r{}_{s[n}Y_{|rm|pq]} + \gamma^r{}_{m[n}Y_{|rs|pq]} + \gamma_{sm}{}^rY_{r[npq]} = 0 \tag{4.9}$$

where $V_{sm[npq]}$ is defined by

$$V_{sm[npq]} = R_{sm[np;q]} - 2R_{smt[n}\gamma^t{}_{pq]} + R_{[pq|tm}\gamma^t{}_{s|n]} - R_{[pq|ts|}\gamma^t{}_{mn]} \tag{4.10}$$

and when written out explicitly [subject of course to the simplifications (3.1)] each independent nontrivial $V_{sm[npq]}$ corresponds to one of the $V_m{}^I$, V^{II}_m in the new notation.

When (4.3, 4) are considered as differential equations in Ψ_0 and Ψ_2, the integrability conditions are found to be

$$V^I_{[m;n]} + W^I_{mn} + \tfrac{1}{2}X_m{}^p{}_n\nabla_p\Psi_2 + 3a_{[m}V_{n]}{}^I + b_{[m}V^{II}_{n]} - \gamma_{[m}{}^p{}_{n]}V_p{}^I = 0 \tag{4.11}$$

$$V^{II}_{[m;n]} + W^{II}_{mn} + \tfrac{1}{2}X_m{}^p{}_n\nabla_p\Psi_2 + \{a_{[m} + 4c_{[m}\}V^{II}_{n]} + 3b_{[m}V_{n]}{}^I - \gamma_{[m}{}^p{}_{n]}V^{II}_p = 0 \tag{4.12}$$

When (4.2) and (4.5, 6) are examined for algebraic consistency the following conditions need to be satisfied:

$$W^I_{12} - W^I_{34} + 3\Psi_2\{Y_{1342} - Y_{2431}\} + \Psi_0\{Y_{2442} - Y_{1331}\} = 0 \tag{4.13a}$$

$$W^I_{13} - W^{II}_{24} + 3\Psi_2\{Y_{1343} - Y_{1321}\} + \Psi_0\{Y_{2443} - Y_{2421} + 2Y_{1224} - 2Y_{3424}\} = 0 \tag{4.13b}$$

$$W^I_{24} - W^{II}_{13} - 3\Psi_2\{Y_{2443} - Y_{2421}\} - \Psi_0\{Y_{1343} - Y_{1321} + 2Y_{1213} - 2Y_{3413}\} = 0 \tag{4.13c}$$

When equations (4.2) and (4.5, 6) are considered as differential equations in a_m, b_m, and c_m it is again found that only one new set of integrability conditions arise [from (4.5, 6) alone] and they are found to be

$$\begin{aligned} W^I_{[mn;p]} &+ \tfrac{3}{2}\Psi_2 X_{[m}{}^s{}_n a_{p];s} + \tfrac{1}{2}\Psi_0 X_{[m}{}^s{}_n b_{p];s} \\ &- 2\gamma_{[m}{}^s{}_n W^I_{p]s} - 3W^I_{[mn}a_{p]} - W^{II}_{[mn}b_{p]} \\ &- V^{II}_{[m}\{b_{n;p]} - \gamma_n{}^s{}_{p]}b_s + 2a_n b_{p]} + 4b_n c_{p]}\} \\ &- 3V^I_{[m}\{a_{n;p]} - \gamma_n{}^s{}_{p]}a_s\} \\ &- \tfrac{3}{2}\Psi_2 Y_{[m}{}^s{}_{np]}a_s - \tfrac{1}{2}\Psi_0 Y_{[m}{}^s{}_{np]}b_s = 0 \end{aligned} \tag{4.14}$$

$$\begin{aligned} W^{II}_{[mn;p]} &+ \tfrac{3}{2}\Psi_2 X_{[m}{}^s{}_n b_{p];s} + \tfrac{1}{2}\Psi_0 X_{[m}{}^s{}_n\{a_{p];s} + 4c_{p];s}\} \\ &- 2\gamma_{[m}{}^s{}_n W^{II}_{p]s} - 3W^I_{[mn}b_{p]} - W^{II}_{[mn}\{a_{p]} + 4c_{p]}\} \\ &- 3V^I_{[m}\{b_{n;p]} - \gamma_n{}^s{}_{p]}b_s + 4c_n b_{p]} + 2b_n a_{p]}\} \\ &- V^{II}_{[m}\{a_{n;p]} - \gamma_n{}^s{}_{p]}a_s + 4c_{n;p]} - 4\gamma_n{}^s{}_{p]}c_s\} \\ &- \tfrac{3}{2}\Psi_2 Y_{[m}{}^s{}_{np]}b_s - \tfrac{1}{2}\Psi_0 Y_{[m}{}^s{}_{np]}\{a_s + 4c_s\} = 0 \end{aligned} \tag{4.15}$$

These five sets of equations (4.8), (4.9), (4.11, 12), (4.13), and (4.14, 15) hold for any values of X_{mnp}, Y_{mnpq}, $V_m{}^I$, V_m^{II}, $W_m{}^I$, W_m^{II} and can therefore be considered as the identities linking the system of equations (3.2), (3.3), (3.7, 8), and (3.9, 10).

There are many different ways in which this very considerable amount of redundancy could be exploited, but in this paper interest will concentrate on the possibility of using the post-Bianchi equations as replacements for other, perhaps less manageable equations, e.g., the set (3.2).

5. SUFFICIENT SUBSYSTEMS OF THE COMPLETELY INTEGRABLE SYSTEM

In Section 3 a completely integrable system of four sets of equations was determined, and in Section 4 it was realized that a considerable amount of redundancy exists within this system, which raises the possibility of choosing compact subsystems which are sufficient to ensure that the complete system is satisfied. [Indeed it is well known—and trivial to deduce from the identities—that equations (3.2) and (3.3) alone form such a sufficient subsystem.]

When all four sets of equations are written out explicitly in N.P. formalism it is clear that the first set of equations (3.2) is of a different nature from the other three sets, (3.3), (3.7, 8), and (3.9, 10); and the anomalous character of this set is bound to raise special difficulties in any integration program for the system. So the question arises whether the set (3.2) is in fact redundant.

An examination of identity (4.11, 12) shows that, providing (3.3), (3.7, 8), and (3.9, 10) hold, then

$$X_m{}^p{}_n \nabla_p \Psi_2 = 0 \tag{5.1a}$$

$$X_m{}^p{}_n \nabla_p \Psi_0 = 0 \tag{5.1b}$$

If it is assumed that only the most general Petrov type I spaces are considered, i.e., those where Ψ_2 and Ψ_0 supply four functionally independent quantities, then (5.1a, b) implies

$$X_m{}^p{}_n = 0 \tag{5.2}$$

(In cases where less than four independent quantities are supplied, a more detailed analysis will be required, starting with a reexamination of the consistency of the system of equations.)

So it can be concluded that the first set of equations need not be considered explicitly, which gives the following result: *The three sets of equations* (3.3), (3.7, 8), *and* (3.9, 10) *form a sufficient subsystem of the completely integrable system.*

It is clear that the above result exploits only a small part of the available redundancy. (It is noted that the result can easily be deduced without using the identities, as such, at all.) The possibility arises of exploiting the redundancy further to establish a still more compact subsystem and in particular a subsystem which can be written concisely and exclusively in the new notation. An examination of the identities reveals that it is impossible to exclude the entire set (3.3) [i.e., to have a subsystem comprising only (3.7, 8) and (3.9, 10)] but it is clearly possible to include only *some* of equations (3.3) alongside (3.7, 8) and (3.9, 10) and yet ensure that the complete system is satisfied. There are many ways such a sufficient subset of (3.3) can be chosen but it would clearly be desirable to be able to choose a subset which can be presented concisely and naturally in the new notation.

When the set of equations (3.3) are considered in detail, it is found that only three equations transfer simply and naturally into the new notation. These are

$$Y_{1342} + Y_{2431} = 0 \tag{5.3a}$$

$$Y_{2442} + Y_{1331} = 0 \tag{5.3b}$$

$$Y_{1212} - Y_{3412} - Y_{1234} + Y_{3434} = 0 \tag{5.3c}$$

which become

$$a^m{}_{;m} - \gamma_m{}^{pm}a_p + a^m a_m - b^m b_m = -2\Psi_2 \tag{5.4a}$$

$$b^m{}_{;m} - \gamma_m{}^{pm}b_p + 4b^m c_m = 2\Psi_0 \tag{5.4b}$$

$$c^m{}_{;m} - \gamma_m{}^{pm}c_p + 2c^m a_m - \tfrac{1}{2}a^m a_m + \tfrac{1}{2}b^m b_m = 2\Psi_2 \tag{5.4c}$$

When these three equations (5.4a, b, c), together with the sets of equations (3.7, 8) and (3.9, 10) are substituted into the identities, it can be shown that

$$X_{mnp} = 0 \qquad \text{for all } m, n, p \tag{5.5a}$$

$$Y_{mnpq} = 0 \qquad \text{for all } m, n, p, q \tag{5.5b}$$

(The details of this calculation are given in Appendix B.)

So the following result has been obtained: *The three sets of equations* (5.4), (3.7, 8), *and* (3.9, 10) *form a sufficient subsystem of the completely integrable system.*

Therefore a remarkably concise and simple system of equations has been determined in the new notation, and the simple structure is even more apparent when the equations are written in ordinary tensor notation. In this presentation the subsystem is given by

$$a^a{}_{;a} + a^a a_a - b^a b_a = -2\Psi_2 \tag{5.6a}$$

$$b^a{}_{;a} + 4b^a c_a = 2\Psi_0 \tag{5.6b}$$

$$c^a{}_{;a} + 2c^a a_a - \tfrac{1}{2}a^a a_a + \tfrac{1}{2}b^a b_a = 2\Psi_2 \tag{5.6c}$$

$$\nabla_a \Psi_2 - 3a_a\Psi_2 - b_a\Psi_0 = 0 \tag{5.7a}$$

$$\nabla_a \Psi_0 - 3b_a\Psi_2 - \{a_a + 4c_a\}\Psi_0 = 0 \tag{5.7b}$$

$$3\Psi_2\{a_{[a;b]}\} + \Psi_0\{b_{[a;b]} + 2a_{[a}b_{b]} + 4b_{[a}c_{b]}\} = 0 \tag{5.8a}$$

$$3\Psi_2\{b_{[a;b]} + 2b_{[a}a_{b]} + 4c_{[a}b_{b]}\} + \Psi_0\{a_{[a;b]} + 4c_{[a;b]}\} = 0 \tag{5.8b}$$

where

$$a_a = -\mu Z_{1a} + \rho Z_{2a} + \pi Z_{3a} - \tau Z_{4a} \tag{5.9a}$$

$$b_a = \sigma Z_{1a} - \lambda Z_{2a} - \kappa Z_{3a} + \nu Z_{4a} \tag{5.9b}$$

$$c_a = \gamma Z_{1a} - \epsilon Z_{2a} - \alpha Z_{3a} + \beta Z_{4a} \tag{5.9c}$$

This new presentation of the vacuum Einstein–Petrov type I equations is not only different from existing presentations, but in some ways the structure, especially the differential structure, is simpler. The natural and simple nature of these results suggests that they could have been obtained more directly, and that there is some underlying structure which still remains to be fully understood and exploited.

From a practical point of view it is important to note that the covariant derivative can always be replaced by the much more manageable ordinary partial derivative [although of course, the determinant g of the metric tensor g_{ab} still occurs in (5.6)]. Further there is an obvious subclass of spaces (when a_a, b_a, and c_a are chosen as gradient vectors) where some of the equations are identically satisfied and the system immediately reduces to only three equations. Some of these spaces are considered in the following section.

6. SIMPLIFICATION OF EQUATIONS FOR A SPECIAL CASE

The system of equations (5.6a, b, c), (5.7a, b), (5.8a, b) are now considered in their own right—the origin of the vectors a_a, b_a, and c_a is no longer important. A solution to this system of equations (by which is meant having solved for a_a, b_a, c_a, a^a, b^a, c^a, Ψ_0, Ψ_2) is sufficient to give a unique metric g_{ab}

which will be a metric for vacuum Petrov type I spaces. An example will illustrate the technique.

The special case,

$$a_a = 2c_a \tag{6.1}$$

is considered. When this specialization is substituted into (5.7) they become

$$\nabla_a \Psi_2 - 3a_a \Psi_2 - b_a \Psi_0 = 0 \tag{6.2a}$$

$$\nabla_a \Psi_0 - 3b_a \Psi_2 - 3a_a \Psi_0 = 0 \tag{6.2b}$$

which are easily rearranged into

$$6a_a = \nabla_a \ln \{3\Psi_2{}^2 - \Psi_0{}^2\} \tag{6.3}$$

$$2(3)^{1/2} b_a = \nabla_a \ln \{(\Psi_0 + 3^{1/2}\Psi_2)/(\Psi_0 - 3^{1/2}\Psi_2)\} \tag{6.4}$$

Since a_a and b_a are clearly gradient vectors, this property, together with condition (6.1), ensures that the post-Bianchi equations (5.8a, b) are identically satisfied.

So only equations (5.6) remain to be satisfied, and under the substitution (6.1) they simplify to

$$b^a b_a = 3\Psi_2 \tag{6.5a}$$

$$a^a{}_{;a} + a^a a_a = \Psi_2 \tag{6.5b}$$

$$b^a{}_{;a} + 2b^a a_a = 2\Psi_0 \tag{6.5c}$$

or,

$$(\nabla b)\cdot(\nabla b) = 3\Psi_2 \tag{6.6a}$$

$$\Box a + (\nabla a)\cdot(\nabla a) = \Psi_2 \tag{6.6b}$$

$$\Box b + 2(\nabla b)\cdot(\nabla a) = 2\Psi_0 \tag{6.6c}$$

where the complex scalars a, b are defined by

$$a_a = a_{,a} \tag{6.7a}$$

$$b_a = b_{,a} \tag{6.7b}$$

Since it has been assumed that Ψ_0 and Ψ_2 are functionally independent, they could be used to provide the four coordinates, but in this case it is more convenient to let the functionally independent scalars a and b provide the coordinates,

$$a = \xi = x_1 + ix_2 \tag{6.8a}$$

$$b = \eta = x_3 + ix_4 \tag{6.8b}$$

In this coordinate system the equations (6.6) become

$$g^{\eta\eta} = 3\Psi_2 \tag{6.9a}$$

$$g^{-1/2}\{g^{1/2}g^{\xi a}\}_{,a} + g^{\xi\xi} = \Psi_2 \tag{6.9b}$$

$$g^{-1/2}\{g^{1/2}g^{\eta a}\}_{,a} + 2g^{\xi\eta} = 2\Psi_0 \tag{6.9c}$$

where

$$3^{1/2}\Psi_2 = ie^{3\xi}\cosh 3^{1/2}\eta \tag{6.10a}$$

$$\Psi_0 = ie^{3\xi}\sinh 3^{1/2}\eta \tag{6.10b}$$

So for this special case, the problem reduces to solving the coupled set of differential equations (6.9) to obtain the metric tensor directly. As would be anticipated, this problem is far from trivial, but it certainly seems possible to establish some general existence criteria, and to extract special solutions by straightforward techniques. The results of such investigations will be given elsewhere.

7. SUMMARY AND DISCUSSION

The original intention of this paper was to establish the completely integrable system of equations discovered by Brans (1977) by a more concise and transparent method. In order to do this it was necessary to look in detail at the structure of the system of equations and not only did this make it easy to obtain Brans' results, but it was possible to round off his work by presenting explicitly the redundancy inherent in this complete system. An immediate consequence of this redundancy is that, for arbitrary Petrov type I spaces, the first set of structure equations can be omitted from the complete system. Since within the N.P. formalism, this particular set of equations is of a different nature (vector rather than scalar) than the other three sets, it is hoped that being able to exclude it will simplify integration procedures within the N.P. formalism.

The conciseness of presentation for both the complete system and the associated identities, was achieved by introducing a new formalism which emphasized the comparatively simple structure of the Bianchi equations for vacuum Petrov type I spaces. This new presentation also revealed that the post-Bianchi equations were of a much simpler structure than might be suspected from their presentation in the N.P. formalism (Brans, 1977). Further, it was almost trivial to deduce that no higher integrability conditions exist; this confirms Brans' suspicion that this result should be obtainable without the detailed computations that he had to carry out.

However, what is felt to be of more significance is that this new formalism, which was introduced originally merely as an aid to computations and presentation, is of importance in its own right. The sufficient subsystem of the complete system is not only very compact but its very natural and simple structure suggests that it could have been derived by more direct means which are not immediately obvious. (It is remarkable that the only three equations of the second set of structure equations which can be written simply in the new notation are precisely the minimum equations needed to combine with the Bianchi and post-Bianchi equations to ensure a sufficient subsystem. It is also interesting that these three equations, probably the crucial ones of the subsystem, have such a standard form.) So although some more insight has been gained into the structure of these spaces, it is still felt, echoing Brans (1977), that there is still much more to be understood.

Finally, of immediate interest is the real possibility of obtaining new exact solutions and also of extracting physically significant information from the "wave equations" in Section 7.

APPENDIX A

The twelve equations (3.9) and (3.10) are given explicitly as

$$\begin{aligned} W^{I}_{12} \equiv{}& 3\Psi_2\{\Delta\rho + D\mu - (\gamma + \bar\gamma)\rho + (\epsilon + \bar\epsilon)\mu + \tau\bar\tau - \pi\bar\pi\} \\ &+ \Psi_0\{-\Delta\lambda - D\sigma + (\bar\gamma - 3\gamma)\lambda + (3\epsilon - \bar\epsilon)\sigma + \kappa\tau \\ &\qquad + \kappa\bar\pi - \nu\bar\tau - \nu\pi + 2\sigma\rho - 2\mu\lambda\} \\ ={}& 0 \end{aligned}$$

$$\begin{aligned} W^{I}_{13} \equiv{}& 3\Psi_2\{\delta\rho - D\tau + (\bar\pi - \bar\alpha - \beta)\rho + (\bar\rho + \epsilon - \bar\epsilon)\tau + \mu\kappa - \sigma\pi\} \\ &+ \Psi_0\{-\delta\lambda + D\nu + (2\tau + 5\beta + \bar\alpha - \bar\pi)\lambda + (\bar\epsilon - 5\epsilon - \bar\rho - 2\rho)\nu\} \\ ={}& 0 \end{aligned}$$

$$\begin{aligned} W^{I}_{14} \equiv{}& 3\Psi_2\{\bar\delta\rho + D\pi - (\alpha + \bar\beta)\rho + (\epsilon - \bar\epsilon)\pi + \tau\bar\sigma + \mu\bar\kappa\} \\ &+ \Psi_0\{-\bar\delta\lambda - D\kappa - (3\pi + 3\alpha - \bar\beta)\lambda + (3\rho + 3\epsilon + \bar\epsilon)\kappa - \nu\bar\sigma - \sigma\bar\kappa\} \\ ={}& 0 \end{aligned}$$

$$\begin{aligned} W^{I}_{23} \equiv{}& 3\Psi_2\{-\delta\mu - \Delta\tau - (\beta + \bar\alpha)\mu + (\gamma - \bar\gamma)\tau + \pi\bar\lambda + \rho\bar\nu\} \\ &+ \Psi_0\{\delta\sigma + \Delta\nu - (3\tau + 3\beta - \bar\alpha)\sigma + (3\mu + 3\gamma + \bar\gamma)\nu - \kappa\bar\lambda - \lambda\bar\nu\} \\ ={}& 0 \end{aligned}$$

$$\begin{aligned} W^{I}_{24} \equiv{}& 3\Psi_2\{-\bar\delta\mu + \Delta\pi + (\bar\tau - \bar\beta - \alpha)\mu + (\bar\mu + \gamma - \bar\gamma)\pi + \rho\nu - \lambda\tau\} \\ &+ \Psi_0\{\delta\sigma - D\kappa + (2\pi + 5\alpha + \bar\beta - \tau)\lambda + (\bar\gamma - 5\gamma - \bar\mu - 2\mu)\kappa\} \\ ={}& 0 \end{aligned}$$

$$\begin{aligned} W^{I}_{34} \equiv{}& 3\Psi_2\{\bar\delta\tau + \delta\pi - (\alpha - \bar\beta)\tau - \rho\bar\mu + \mu\bar\rho + (\beta - \bar\alpha)\pi\} \\ &+ \Psi_0\{-\bar\delta\nu + \delta\kappa - (\bar\beta + 3\alpha)\nu + (3\beta + \bar\alpha)\kappa + \sigma(\rho - \bar\rho) \\ &\qquad + \lambda(\bar\mu - \mu) + 2\tau\kappa - 2\pi\nu\} \\ ={}& 0 \end{aligned}$$

$$\begin{aligned} W^{II}_{12} \equiv{}& 3\Psi_2\{-\Delta\lambda - D\sigma + \lambda(\bar\gamma + 5\gamma) - (5\epsilon + \bar\epsilon)\sigma + \kappa\bar\pi - \nu\pi \\ &\qquad + \kappa\tau - \nu\bar\tau - 2\rho\sigma + 2\mu\lambda\} \\ &+ \Psi_0\{\Delta\rho + D\mu - \rho(\gamma + \bar\gamma) + \tau\bar\tau + \mu(\epsilon + \bar\epsilon) - \pi\bar\pi \\ &\qquad - 4(\Delta\epsilon + D\gamma + 2\epsilon\gamma - \bar\gamma\epsilon - \gamma\bar\epsilon - \alpha\tau + \beta\pi + \beta\bar\tau - \alpha\bar\pi)\} \\ ={}& 0 \end{aligned}$$

$$\begin{aligned}
W^{II}_{13} \equiv\ & 3\Psi_2\{-\delta\lambda + D\nu - (2\tau + 3\beta - \bar{\alpha} + \pi)\lambda + (\bar{\epsilon} + 3\epsilon - \bar{\rho} + 2\rho)\nu\} \\
& + \Psi_0\{\delta\rho - D\tau + (\bar{\pi} - \bar{\alpha} - \beta)\rho + (\bar{\rho} + \epsilon - \bar{\epsilon})\tau + \mu\kappa - \sigma\pi \\
& \quad - 4(\delta\epsilon - D\beta - \alpha\sigma + \beta\bar{\rho} + \kappa\gamma + \bar{\pi}\epsilon - \beta\bar{\epsilon} - \bar{\alpha}\epsilon)\} \\
=\ & 0
\end{aligned}$$

$$\begin{aligned}
W^{II}_{14} \equiv\ & 3\Psi_2\{-\bar{\delta}\lambda - D\kappa + \lambda(\bar{\beta} + 5\alpha) + \kappa(\bar{\epsilon} - 5\epsilon) - \rho\kappa + \lambda\pi - \nu\bar{\sigma} - \sigma\bar{\kappa}\} \\
& + \Psi_0\{\bar{\delta}\rho + D\pi - (\alpha + \bar{\beta})\rho + (\epsilon - \bar{\epsilon})\pi + \tau\bar{\sigma} + \mu\bar{\kappa} \\
& \quad - 4(\bar{\delta}\epsilon + D\alpha - \alpha\bar{\epsilon} - \bar{\beta}\epsilon - \alpha\rho + \beta\bar{\sigma} + \gamma\bar{\kappa} + \epsilon\pi \\
=\ & 0
\end{aligned}$$

$$\begin{aligned}
W^{II}_{23} \equiv\ & 3\Psi_2\{\delta\sigma + \Delta\nu + \sigma(\bar{\alpha} + 5\beta) + \nu(\bar{\gamma} - 5\gamma) - \mu\nu + \sigma\tau - \kappa\bar{\lambda} - \lambda\bar{\nu}\} \\
& + \Psi_0\{-\delta\mu - \Delta\tau - (\beta + \bar{\alpha})\mu + (\gamma - \bar{\gamma})\tau + \pi\bar{\lambda} + \rho\bar{\nu} \\
& \quad - 4(-\delta\gamma - \Delta\beta - \beta\tau - \bar{\alpha}\gamma - \beta\mu + \alpha\bar{\lambda} + \epsilon\bar{\nu} + \gamma\lambda + \beta\gamma - \beta\bar{\gamma})\} \\
=\ & 0
\end{aligned}$$

$$\begin{aligned}
W^{II}_{24} \equiv\ & 3\Psi_2\{\bar{\delta}\sigma - \Delta\kappa - (2\pi + 3\alpha - \bar{\beta} + \tau)\sigma + (\bar{\gamma} + 3\gamma - \bar{\mu} + 2\mu)\kappa\} \\
& + \Psi_0\{-\bar{\delta}\mu + \Delta\pi + (\bar{\tau} - \bar{\beta} - \alpha)\rho + (\bar{\mu} + \gamma - \bar{\gamma})\pi + \rho\nu - \lambda\tau \\
& \quad - 4(-\bar{\delta}\gamma + \Delta\alpha - \lambda\beta + \bar{\mu}\alpha + \nu\epsilon + \bar{\tau}\gamma - \alpha\bar{\gamma} - \bar{\beta}\gamma)\} \\
=\ & 0
\end{aligned}$$

$$\begin{aligned}
W^{II}_{34} \equiv\ & 3\Psi_2\{-\bar{\delta}\nu - \delta\kappa + \nu(\bar{\beta} - 5\alpha) - \kappa(5\beta - \bar{\alpha}) - \sigma\mu - \lambda\mu - \sigma\rho \\
& \quad + \lambda\bar{\mu} - 2\tau\kappa + 2\pi\nu\} \\
& + \Psi_0\{\bar{\delta}\tau + \delta\pi - \tau(\alpha - \bar{\beta}) - \rho\bar{\mu} + \pi(\beta - \bar{\alpha}) + \mu\bar{\rho} \\
& \quad - 4(\bar{\delta}\beta + \delta\alpha + 2\alpha\beta + \beta\bar{\beta} - \alpha\bar{\alpha} - \gamma\rho + \epsilon\mu - \epsilon\bar{\mu} + \gamma\bar{\rho})\} \\
=\ & 0
\end{aligned}$$

As noted in Sections 3 and 4, three of the above equations are not independent when considered together with the set (3.3′). When the above 12 equations are compared with the nine equations given by Brans (1977), it is seen that the three dependent equations omitted by Brans, are W^{I}_{14}, W^{II}_{24}, W^{I}_{34}. Further it is easy to see that the four equations W^{I}_{12}, W^{I}_{13}, W^{I}_{14}, W^{I}_{23}, agree identically with four equations from Brans' set, but that the remaining five equations have to be combined with appropriate equations from the set (3.3′) to agree identically with the remaining five in Brans' set. (It should be noted that there are some minor disagreements between the two sets, due it is suspected to misprints in Brans' paper.)

APPENDIX B

In order to obtain the required result it will be necessary first to establish the following lemma.

Lemma. Consider a tensor T_{abcd} with the following properties:

(a) $$T_{abcd} = T_{ab[cd]} = T_{[ab]cd} = T_{cdab}$$

(b) $$T_{a[bcd]} = 0$$

(c) $$T^{ab}{}_{cd} = \tilde{T}^{ab}{}_{cd} + 2\delta^{[a}{}_{[c}T^{]b}{}_{d]} - \tfrac{1}{3}\delta^{[a}{}_{[c}\delta^{b]}{}_{d]}T$$

where $T^a{}_c = T^{ab}{}_{cb}$, $T = T^a{}_a$

(d) $$T_{ab[cd;e]} = 0$$

In Einstein–Petrov type I vacuum spaces the conditions

(i) $\tilde{T}_{abcd} = 0$
(ii) $T = 0$

are sufficient to ensure

$$T_{abcd} = 0$$

Proof. When conditions (i) and (ii) are substituted, via the decomposition formula into property (d), it is found that

$$T_{a[b;c]} = 0 \tag{B.1}$$

The integrability conditions for these equations are

$$T_{a[b;cd]} = 0 \tag{B.2}$$

or

$$R_a{}^e{}_{[cd}T_{b]e} = 0 \tag{B.3}$$

When each individual equation of (B.3) is written out explicitly for vacuum Petrov type I spaces, it is clear that

$$T_{ab} = 0 \tag{B.4}$$

and hence the lemma is proven. ■

Turning now to the proof of the main result. When equations (5.4), (3.7, 8), and (3.9, 10) are substituted into the identities (4.8)–(4.15) the following results are obtained, with the help of the decomposition formula:

$$X_{mnp} = 0 \tag{B.5}$$

$$Y = 0 \tag{B.6}$$

$$\tilde{Y}_{mnpq} = 0 \tag{B.7}$$

By applying the tetrad version of the above lemma it follows immediately that

$$Y_{mnpq} = 0 \tag{B.8}$$

which is the required result.

NOTE ADDED IN PROOF

It has been realized that the system of equations (5.6), (5.7), (5.8), and (5.9) do not follow as easily from the system (5.4), (3.7, 8), and (3.9, 10) as implied. The detailed justification for this particular result will be given in a separate paper.

REFERENCES

Brans, C. H. (1977). *Journal of Mathematical Physics*, **18**, 1378.

Edgar, S. B. (1978). "The structure of tetrad formalisms in General Relativity; the general case." Preprint, to be published in *General Relativity and Gravitation*.

Geroch, R., Held, A., and Penrose, R. (1973). *Journal of Mathematical Physics*, **14**, 874.

Newman, E. T., and Penrose, R. (1962). *Journal of Mathematical Physics*, **3**, 566.

Papapetrou, A. (1970). *Annals Institut Henri Poincaré*, **13**, 271.

Papapetrou, A. (1971a). *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, **272**, 1537.

Papapetrou, A. (1971b). *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, **272**, 1613.